

On matter coupled to the higher spin square

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ABSTRACT: Gaberdiel and Gopakumar recently proposed that the tensionless limit of string theory on $AdS_3 \times S^3 \times T^4$ takes the form of a higher spin theory with a gauge algebra that is referred to as the higher spin square. In this note, we formulate the linearized Vasiliev-type equations which describe a matter field coupled to the higher spin square. We study the particle spectrum of this field and show that it accounts for the entire untwisted sector of the dual symmetric orbifold CFT, thereby confirming a conjecture by Gaberdiel and Gopakumar. In doing so, we pinpoint the group-theoretic data which determine the spectrum of a matter field coupled to a general higher spin algebra, which we illustrate by revisiting the theory based on the $hs[1/2]$ algebra.

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1 Introduction

The idea that string theory may possess a phase of enhanced higher spin gauge symmetry is a longstanding one [1]. String field theory would most likely take its simplest form when formulated around the vacuum of maximal unbroken symmetry, where it would be maximally constrained by an underlying gauge principle. Phases with massive higher spin fields, such as the standard formulation around the Minkowski background, would then arise as Higgsed phases upon expanding around a vacuum which breaks the higher spin symmetry. The work of Vasiliev (see e.g. [2], [3] for reviews) has shown that interacting higher spin gauge theories allow for an anti-de Sitter vacuum, and it is therefore natural to look for the most symmetric phase of string theory in an AdS compactification.

Though conceptually appealing, these ideas didn't receive a concrete realization until the recent work of Gaberdiel and Gopakumar [4],[5],[6] (see also the related works [7],[8],[9]). They proposed the tensionless limit of string theory on $AdS_3 \times S^3 \times T^4$ as a candidate for the phase with maximal unbroken symmetry. It has a dual description in terms of a two-dimensional superconformal field theory: the N -fold symmetric product orbifold of the $\mathcal{N} = 4$ SCFT on the torus T^4 , denoted by $Sym^N(T^4)$, in the limit of large N . From analyzing the dual theory they were able to isolate the symmetry algebra which plays the role of the gauge algebra in the bulk. This algebra, which contains exponentially more generators than the standard higher spin algebras, is nevertheless fully determined by the action of standard 'horizontal' and 'vertical' higher spin algebras and therefore goes under the name of the higher spin square (hss) [5]. We will review its definition in section 4 below.

Furthermore, the spectrum of the untwisted sector of $Sym^N(T^4)$ has a simple content in terms of hss representations.

In this note, we will take a first step towards reformulating tensionless string theory in the bulk as a higher spin gauge theory with hss symmetry. In particular, we will show that the entire spectrum of the untwisted sector arises, on the bulk side, from a single matter field coupled to the hss gauge fields, thereby confirming a conjecture made in [6]. Along the way, we will single out the group-theoretic data which determine the spectrum of single-particle states described by Vasiliev-like matter equations in AdS_3 [10] for a general higher spin algebra. We will illustrate this by revisiting the Vasiliev theory based on the higher spin algebra $hs[1/2]$ from this point of view.

Let us first collect some of the results of [6]. It was shown there that the NS-NS sector partition function of the untwisted sector of $Sym^N(T^4)$ can, at large N , be written a way which is suggestive of a dual bulk interpretation:

$$Z_U(q, \bar{q}, y, \bar{y}) \equiv \text{tr}_U q^{L_0} \bar{q}^{\bar{L}_0} y^{2J_0^3} \bar{y}^{2\bar{J}_0^3} \quad (1.1)$$

$$= Z_{hss}^{gauge}(q, \bar{q}, y, \bar{y}) Z_{hss}^{matter}(q, \bar{q}, y, \bar{y}). \quad (1.2)$$

Here, Z_{hss} is the mod-squared of the vacuum character of the chiral algebra of the dual CFT

$$Z_{hss}^{gauge}(q, \bar{q}, y, \bar{y}) = |Z_{vac}(q, y)|^2. \quad (1.3)$$

In analogy with the Vasiliev higher spin theories [12], this contribution is expected to come from boundary excitations of gauge fields taking values in the higher spin square Lie algebra hss . The second factor in (1.2) is suggestive of a contribution from matter in the bulk. It takes the ‘multi-particle’ form¹

$$Z_{hss}^{matter}(q, \bar{q}, y, \bar{y}) = \exp \sum_{n=1}^{\infty} \frac{Z_{hss}^{1-part}(q^n, \bar{q}^n, (-1)^{n+1} y^n, (-1)^{n+1} \bar{y}^n)}{n} \quad (1.4)$$

and appears to describe multi-particle excitations of a theory whose single particle spectrum is given by

$$Z_{hss}^{1-part}(q, \bar{q}, y, \bar{y}) = |\chi_{min}(q, y)|^2 \quad (1.5)$$

Here, χ_{min} is the character of the so-called minimal representation [4] of the higher spin square. More explicitly, it is given by

$$\chi_{min}(q, y) = Z_{T^4}(q, y) - 1 \quad (1.6)$$

where Z_{T^4} is the chiral part of the partition function of the SCFT on T^4 , i.e. the partition function of 4 real chiral bosons and 4 Majorana-Weyl fermions:

$$Z_{T^4}(q, y) = \prod_{n=1}^{\infty} \frac{(1 + yq^{n-\frac{1}{2}})^2 (1 + y^{-1}q^{n-\frac{1}{2}})^2}{(1 - q^n)^4}. \quad (1.7)$$

Our goal in this note will be to show how (1.5) arises from single-particle excitations of matter fields in the bulk.

¹The minus signs in this expression account for statistics, as we will explain in more detail in section 4 below.

2 Free Vasiliev system and single particle states

In this section we will review the linearized equations describing matter coupled to massless higher spin fields in 2+1 dimensions. These equations were originally [10] written down for the higher spin theories with higher spin algebra $hs[\lambda]$ (and their supersymmetric extensions), but we will not yet specify the higher spin algebra here, emphasizing instead the algebraic ingredients necessary to write down a consistent set of equations. This will pave the way for analyzing matter fields coupled to the higher spin square in section 4.

2.1 Massless higher spin fields

Massless higher spin fields in 2+1 dimensions are described by a gauge theory based on two copies² of a higher spin Lie algebra \mathfrak{h} , with \mathfrak{h} -valued gauge potentials A, \tilde{A} . We will assume \mathfrak{h} to contain an $sl(2, \mathbb{R})$ subalgebra, which we will single out as the subsector which describes Einstein gravity in AdS_3 . In general, the spin content of the theory is determined by the decomposition of the adjoint representation of \mathfrak{h} into representations of this $sl(2, \mathbb{R})$ subalgebra. The equations of motion state that A, \tilde{A} are flat:

$$dA - A \wedge A = 0, \quad d\tilde{A} - \tilde{A} \wedge \tilde{A} = 0. \quad (2.1)$$

Hence the gauge sector doesn't contain any local propagating degrees of freedom, although a careful treatment of the boundary conditions at infinity [13],[14] generically uncovers the existence of boundary excitations.

The equations of motion are formally invariant under finite higher spin gauge transformations of the form

$$A \rightarrow hAh^{-1} + dh h^{-1}, \quad \tilde{A} \rightarrow \tilde{h}\tilde{A}\tilde{h}^{-1} + d\tilde{h}\tilde{h}^{-1} \quad (2.2)$$

where h, \tilde{h} belong to H , the set of formal exponentials of elements of \mathfrak{h} . We will not address here the question for which elements of \mathfrak{h} the exponential is well-defined, nor whether H can be given the structure of a Lie group. In the case where \mathfrak{h} is the higher spin algebra $hs[\lambda]$, these issues were addressed in [15].

2.2 Linearized matter equations

Next we review the linearized equations describing matter in a higher spin background specified by A, \tilde{A} . In most known examples, the higher spin Lie algebra \mathfrak{h} can be embedded in an associative algebra \mathfrak{a} , such that the Lie bracket in \mathfrak{h} arises from the commutator in \mathfrak{a} . If this is the case, we can write down linearized equations³ for two scalar matter fields

²See however [11] for an example of a higher spin theory based on a Lie algebra which is simple, rather than a product of two isomorphic copies.

³In the original work [10], see also [16], the equations were written more succinctly by introducing an extra Grassmann element ϕ , satisfying $\phi^2 = 1$, and the associated projection operators $P_{\pm} = \frac{1 \pm \phi}{2}$. Combining the fields as $W = AP_+ + \tilde{A}P_-$, $B = CP_+ + \tilde{C}P_-$, the equations (2.1,2.3) reduce to

$$dW - W \wedge W = 0, \quad dB - WB + B\pi(W) = 0$$

where the operation π sends $\phi \rightarrow -\phi$.

C, \tilde{C} taking values in the associative algebra \mathfrak{a} :

$$dC - AC + C\tilde{A} = 0 \qquad d\tilde{C} - \tilde{A}\tilde{C} + \tilde{C}A = 0. \quad (2.3)$$

Here the fields are multiplied using the associative product in \mathfrak{a} . Note that the consistency of (2.3) is guaranteed by (2.1). These equations are usually written in an equivalent ‘star-product’ form, in which the \mathfrak{a} -valued fields are replaced by c -number functions (or ‘symbols’) by using a specific operator ordering prescription, and the effect of the operator product is captured by a suitably defined star-product. We will however keep working with \mathfrak{a} -valued fields in this note. The equations (2.3) are higher spin gauge invariant with the fields transforming as

$$C \rightarrow hC\tilde{h}^{-1}, \qquad \tilde{C} \rightarrow \tilde{h}\tilde{C}h^{-1} \quad (2.4)$$

When \mathfrak{h} is one of the standard $hs[\lambda]$ higher spin algebras and \mathfrak{a} is the underlying ‘lone-star product’ associative algebra [17], the equations (2.3), when expanded around the AdS background, describe the unfolded form of the Klein-Gordon equation for a scalar field of mass squared $\lambda^2 - 1$. We will review this in some detail for the case of $\lambda = \frac{1}{2}$ in section 3 below.

One might think that it is consistent to truncate the theory to keep only the C or \tilde{C} matter field. We will see however in the next section that C describes only negative frequency modes, while \tilde{C} describes positive frequency ones. In order to have a sensible phase space lending itself to quantization, we should therefore keep both C and \tilde{C} even at the linearized level.

2.3 The AdS background

The advantage of the unfolded form of the equations is that we can easily write down the general solution to (2.3) in any background [10]. Indeed, writing the flat connections A, \tilde{A} locally in a pure gauge form,

$$A = dg g^{-1}, \qquad \tilde{A} = d\tilde{g} \tilde{g}^{-1} \quad (2.5)$$

we can transform to a gauge where $A = \tilde{A} = 0$. In this gauge the solutions for the matter fields are simply constant elements $C^0, \tilde{C}^0 \in \mathfrak{a}$. Introducing a basis $\{e_a\}_a$ for \mathfrak{a} and gauge-transforming back, we can write the general solution to (2.3) as a linear combination of the solutions

$$C_a = g e_a \tilde{g}^{-1}, \qquad \tilde{C}_a = \tilde{g} e_a g^{-1}. \quad (2.6)$$

Let us now specialize to the global AdS background, for which g is an exponential of elements in the $sl(2, \mathbb{R})$ subalgebra of \mathfrak{h} with commutation relations $[L_m, L_n] = (m - n)L_{m+n}$ for $m, n = 0, \pm 1$. Separating out the dependence on the AdS radial coordinate ρ , the group elements can be written conveniently as [18]

$$g = R(\rho) e^{iL_0 x_+}, \qquad \tilde{g} = \tilde{R}(\rho) e^{-iL_0 x_-}, \quad (2.7)$$

where $x_{\pm} = t \pm \phi$ and

$$R(\rho) = e^{-\rho L_0} M^{-1}, \quad \tilde{R}(\rho) = e^{\rho L_0} M^{-1}, \quad M = e^{-\frac{i\pi}{4}(L_1 - L_{-1})}. \quad (2.8)$$

The presence of the constant element M is for later convenience: it implements a change of basis which diagonalizes the element $\frac{1}{2}(L_1 + L_{-1})$, in the sense that

$$M \cdot \frac{1}{2}(L_1 + L_{-1}) \cdot M^{-1} = -iL_0. \quad (2.9)$$

Plugging this decomposition into (2.6), the mode solutions for C and \tilde{C} around AdS are given by

$$C_a = R e^{iL_0 x_+} e_a e^{iL_0 x_-} \tilde{R}^{-1}, \quad \tilde{C}_a = \tilde{R} e^{-iL_0 x_-} e_a e^{-iL_0 x_+} R^{-1}. \quad (2.10)$$

2.4 Higher spin representations and single-particle spectrum

We would also like to determine how the solutions (2.10) transform under the global higher spin symmetries of the AdS background. The unfolded formulation makes symmetries manifest in general (see e.g. [19]), and we will now argue that the symmetry properties of the solutions (2.10) follow easily once a certain group-theoretic fact about the algebra \mathfrak{a} is known.

First let us discuss the global symmetries of the AdS background and their action on the matter fields [10],[18]. As before it is convenient to transform to the gauge $A = \tilde{A} = 0$ where the matter fields are constant. Global symmetries are gauge transformations leaving the background $A = \tilde{A} = 0$ invariant; from (2.2) these are generated by constant elements h_0, \tilde{h}_0 and hence the global symmetry consists of two copies of H , which we will denote as $H \times \tilde{H}$. These global symmetries act on constant matter solutions $C^0, \tilde{C}^0 \in \mathfrak{a}$ by left- and right multiplication according to (2.4):

$$C^0 \rightarrow h_0 C^0 \tilde{h}_0^{-1}, \quad \tilde{C}^0 \rightarrow \tilde{h}_0 \tilde{C}^0 h_0^{-1}, \quad h_0 \in H, \tilde{h}_0 \in \tilde{H}. \quad (2.11)$$

We can view the algebra \mathfrak{a} as a representation space for $H \times \tilde{H}$ with, say, \tilde{H} acting from the left and H acting from the right. Therefore we expect to be able to decompose \mathfrak{a} into irreducible $H \times \tilde{H}$ representations as follows

$$\mathfrak{a} = \oplus_i (V_i, W_i). \quad (2.12)$$

This decomposition determines how the space of matter solutions decomposes into irreducible representations of the global symmetry in the gauge $A = \tilde{A} = 0$.

These observations can then be simply gauge transformed to the ‘AdS gauge’ where

$$A = dg g^{-1}, \quad \tilde{A} = d\tilde{g} \tilde{g}^{-1} \quad (2.13)$$

with g, \tilde{g} given in (2.7). The elements which implement the global $H \times \tilde{H}$ symmetry in this gauge are simply obtained from (2.11) by conjugation with g and \tilde{g} :

$$h = g h_0 g^{-1}, \quad \tilde{h} = \tilde{g} \tilde{h}_0 \tilde{g}^{-1}. \quad (2.14)$$

Therefore the decomposition (2.12) also determines the quantum numbers carried by solutions in the AdS gauge (2.13). As a check on our analysis so far, let us work out the infinitesimal gauge parameters implementing the $SL(2, \mathbb{R}) \times \widetilde{SL}(2, \mathbb{R})$ subgroup of the global symmetry:

$$\begin{aligned} \epsilon_{L_0} &= b^{-1} \frac{i}{2} (L_1 + L_{-1}) b, & \epsilon_{L_{\pm 1}} &= b^{-1} e^{\mp i x_+} \left(i L_0 \pm \frac{1}{2} (L_1 - L_{-1}) \right) b \\ \tilde{\epsilon}_{\tilde{L}_0} &= b \frac{i}{2} (L_1 + L_{-1}) b^{-1}, & \tilde{\epsilon}_{\tilde{L}_{\pm 1}} &= b e^{\pm i x_-} \left(i L_0 \pm \frac{1}{2} (L_1 - L_{-1}) \right) b^{-1} \end{aligned} \quad (2.15)$$

where $b = e^{\rho L_0}$. These agree, modulo differences in conventions, with the expressions derived in [20] in the AdS gauge (2.13).

We have argued that the decomposition (2.12) determines how the solution space to (2.3) decomposes into representations of the global symmetry. In free field theory, the space of solutions generally decomposes into positive and negative frequency modes, and only one of these subspaces (usually chosen to be the positive frequency subspace) can be given the structure of a Hilbert space representing the single particle states of the theory. For example, for a complex scalar field, the Hilbert space inner product is constructed from the conserved $U(1)$ current, and a higher spin invariant conserved current also exists for the linearized Vasiliev system based on the $hs[\lambda]$ algebra [21] (see also [22]). It would be interesting to generalize this construction to the higher spin square theory, but in what follows we will simply assume the usual correspondence between positive frequency modes and single particle states.

From (2.10), we see that, if the representations entering in the decomposition (2.12) are unitary (as will be the case in our examples), so that L_0 has positive eigenvalues, the positive frequency modes are contained in the field \tilde{C} , while C describes the negative frequency ones. Therefore only the modes of \tilde{C} represent single particle states, leading to a partition function of the form

$$Z^{1-part}(q, \bar{q}) = \text{tr}_{\mathfrak{a}} q^{L_0} \bar{q}^{\tilde{L}_0} \quad (2.16)$$

$$= \sum_i \chi_{V_i}(q) \chi_{W_i}(\bar{q}), \quad (2.17)$$

where L_0 and \tilde{L}_0 denote quantum numbers under right- and left action respectively. In the last line we have used (2.12) to write the result in terms of characters χ_{V_i} of the representations V_i . A similar formula holds if we refine the partition function with chemical potentials.

To summarize, we have argued that the decomposition (2.12) contains the group-theoretic data which determine the matter spectrum for generic higher spin algebras. In the next two sections we will work out (2.12), first for the warm-up example of the theory based on the $hs[1/2]$ algebra, and subsequently for the case of interest where \mathfrak{h} is the higher spin square hss . Before doing so it may be instructive to work out (2.12) for some finite-dimensional examples. Let \mathfrak{a} the algebra of real 5×5 matrices with the standard matrix multiplication. Let \mathfrak{h} be the subalgebra $sl(2, \mathbb{R})$ whose generators are embedded as

$L_m = L_m^{(2)} \oplus L_m^{(3)}$, with $L_m^{(n)}$ the generators in the n -dimensional representation. It's easy to see that \mathfrak{a} decomposes into $SL(2, \mathbb{R}) \times \widetilde{SL(2, \mathbb{R})}$ representations as

$$\mathfrak{a} = (2, 2) \oplus (2, 3) \oplus (3, 2) \oplus (3, 3). \quad (2.18)$$

If, on the other hand, \mathfrak{h} is $sl(2, \mathbb{R})$ embedded as $L_m = L_m^{(5)}$, the decomposition is simply

$$\mathfrak{a} = (5, 5). \quad (2.19)$$

3 Matter coupled to $hs[1/2]$

As a warmup to the higher spin square case, let us discuss the decomposition (2.12) in the case where \mathfrak{h} is the Lie algebra $hs[1/2]$ and \mathfrak{a} is the underlying ‘lone-star algebra’ $\mathfrak{a}_{hs[1/2]}$ introduced in [17]. This example is instructive since, as will be the case for the higher spin square, $\mathfrak{a}_{hs[1/2]}$ has a realization in terms of undeformed harmonic oscillators. Furthermore, we will be able to give a clearer physical justification of one of the steps involved in this simplified setting.

We start from a single harmonic oscillator

$$[a, a^\dagger] = 1. \quad (3.1)$$

Here, a and a^\dagger can be thought of as related to the Vasiliev operators y_1, y_2 as $a = \frac{y_1}{\sqrt{2i}}, a^\dagger = \frac{y_2}{\sqrt{2i}}$. The standard basis of $\mathfrak{a}_{hs[1/2]}$ is formed by Weyl-ordered monomials of even degree in the oscillators,

$$V_m^s = 2^{1-s} \left[(a^\dagger)^{s-m-1} a^{s+m-1} \right]_W, \quad s \geq 1, |m| < s. \quad (3.2)$$

where the subscript stands for Weyl-ordering. In particular, V_0^1 is the identity operator, and the remaining generators V_m^s with $s \geq 2$ generate, through their commutators, the Lie algebra $hs[1/2]$. It was shown in [16] that multiplying these basis elements and using the commutation relation (3.1) to once again Weyl order the result reproduces the lone-star product of $\mathfrak{a}_{hs[1/2]}$ defined in [17]. The $s = 2$ generators form an $sl(2, \mathbb{R})$ subalgebra, with $L_0 = V_0^2$, $L_{\pm 1} = V_{\pm 1}^2$. One can check that the quadratic Casimir C_2 of this $sl(2, \mathbb{R})$ subalgebra takes the value

$$C_2 = L_0^2 - \frac{1}{2} (L_1 L_{-1} + L_{-1} L_1) = -\frac{3}{16}. \quad (3.3)$$

This observation allows one to alternatively view $\mathfrak{a}_{hs[1/2]}$ as the quotient of the universal enveloping algebra $U(sl(2, \mathbb{R}))$ by the ideal $C_2 + 3/16$.

The basis (3.2) of Weyl-ordered monomials is not very suitable for figuring out the desired decomposition of the form (2.12). For this purpose, one would like to diagonalize the action of the Cartan generators V_0^s both from the left and the right, while one easily checks that this is not the case in the basis (3.2). To remedy this we will instead expand operators in a Fock basis⁴, in terms of basis elements

$$|m\rangle \langle n| = \frac{(a^\dagger)^m}{\sqrt{m!}} |0\rangle \langle 0| \frac{a^n}{\sqrt{n!}} \equiv \frac{1}{\sqrt{m!n!}} \left[(a^\dagger)^m a^n \right]_F. \quad (3.4)$$

⁴A similar basis change in the case of four-dimensional higher spin theories was advocated in [23].

In the last equality we have defined, for later convenience, a ‘Fock-ordering’ acting on oscillator monomials by letting the creation and annihilation operators act on $|0\rangle\langle 0|$ from the left and the right respectively. The relations between Weyl-ordered and Fock-ordered monomials is summarized by the following formulae, whose origin is explained in Appendix A:

$$\left[(a^\dagger)^m a^n\right]_W = \begin{cases} 2^{-n} n! \left[(a^\dagger)^{m-n} L_n^{m-n} (-2aa^\dagger) e^{aa^\dagger}\right]_F & \text{for } m \geq n \\ 2^{-m} m! \left[a^{n-m} L_m^{n-m} (-2aa^\dagger) e^{aa^\dagger}\right]_F & \text{for } n \geq m \end{cases} \quad (3.5)$$

$$\left[(a^\dagger)^m a^n\right]_F = \begin{cases} (-1)^n 2^{m-n+1} n! \left[(a^\dagger)^{m-n} L_n^{m-n} (4aa^\dagger) e^{-2aa^\dagger}\right]_W & \text{for } m \geq n \\ (-1)^m 2^{n-m+1} m! \left[a^{n-m} L_m^{n-m} (4aa^\dagger) e^{-2aa^\dagger}\right]_W & \text{for } n \geq m \end{cases}, \quad (3.6)$$

where L_n^k are the associated Laguerre polynomials. Since the right-hand side of these equations is non-polynomial, one might question the validity of transforming to the Fock basis, and we will provide a physical justification for this step below.

From these expressions it follows that even Weyl-ordered monomials are expressed in terms of even Fock-ordered monomials and vice versa. Therefore (3.5,3.6) allow us to represent the standard basis elements V_m^s of $\mathfrak{a}_{hs[1/2]}$ in terms of the subset of Fock basis elements

$$|m\rangle\langle n| \text{ with } m+n \text{ even}, \quad (3.7)$$

and vice versa. For example one finds, from (3.5),

$$1 = V_0^1 = \sum_n |n\rangle\langle n| \quad (3.8)$$

$$L_0 = V_0^2 = \frac{1}{2} \sum_n \left(n + \frac{1}{2}\right) |n\rangle\langle n| \quad (3.9)$$

$$L_1 = V_1^2 = \frac{1}{2} \sum_n \sqrt{(n+1)(n+2)} |n\rangle\langle n+2| \quad (3.10)$$

$$L_{-1} = V_{-1}^2 = \frac{1}{2} \sum_n \sqrt{(n+1)(n+2)} |n+2\rangle\langle n|. \quad (3.11)$$

Using the Fock basis of $\mathfrak{a}_{hs[1/2]}$ we can now easily derive the decomposition of the form (2.12). Decomposing the harmonic oscillator Fock space \mathcal{F} into

$$\mathcal{F} = \mathcal{F}_{\text{even}} \oplus \mathcal{F}_{\text{odd}}, \quad (3.12)$$

where $\mathcal{F}_{\text{even}}$ (\mathcal{F}_{odd}) are spanned by the basis states with even (odd) excitation number $|2m\rangle$ ($|2m+1\rangle$) respectively, we have, due to (3.7),

$$\mathfrak{a}_{hs[1/2]} = (\mathcal{F}_{\text{even}} \otimes \mathcal{F}_{\text{even}}^*) \oplus (\mathcal{F}_{\text{odd}} \otimes \mathcal{F}_{\text{odd}}^*). \quad (3.13)$$

where $*$ denotes the dual vector space. The subspaces $\mathcal{F}_{\text{even}}$ and \mathcal{F}_{odd} each form irreducible lowest weight representations of $hs[1/2]$ whose lowest weight vectors are $|0\rangle$ and $|1\rangle$ respectively. Indeed, from (3.10) we see that they are annihilated by L_1 , and that their

L_0 eigenvalues are $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Using the description of $hs[1/2]$ as a quotient of the universal enveloping algebra of $sl(2, \mathbb{R})$ by the relation (3.3), one can show that $|0\rangle$ and $|1\rangle$ are also lowest weight vectors under the full $hs[1/2]$ algebra. The corresponding irreducible representations were denoted by ϕ_- and ϕ_+ respectively in [20]. We also note that writing the $hs[1/2]$ elements in the Fock basis as in (3.11) and restricting their action to either \mathcal{F}_{even} or \mathcal{F}_{odd} , we obtain the infinite-dimensional matrix representations of $hs[1/2]$ discussed in [24],[25],[26].

It follows that, for $hs[1/2]$, the decomposition (2.12) reads,

$$\mathfrak{a}_{hs[1/2]} = \left(V_{\phi_-}, V_{\phi_-}^* \right) \oplus \left(V_{\phi_+}, V_{\phi_+}^* \right). \quad (3.14)$$

where $*$ denotes the dual representation. Applying (2.17), the spectrum of single particle states described by the linearized Vasiliev equations is

$$Z_{hs[1/2]}^{1-part} = |\chi_{\phi_-}|^2 + |\chi_{\phi_+}|^2. \quad (3.15)$$

This agrees with the conclusions reached in [10] using somewhat different methods.

Let us now discuss in more detail the explicit solutions which furnish the representations in the decomposition (2.12). These will turn out to be physically sensible, which provides additional justification for the initial step of expanding in the Fock basis (3.4). From (2.10), the first term in (3.14) is furnished by the positive frequency solutions of the form

$$\tilde{C}_{2m,2n}^{AdS} = e^{-i((m+n+\frac{1}{2})t+(n-m)\phi)} \tilde{R} |2m\rangle \langle 2n| R^{-1} \quad (3.16)$$

while the second term is realized on the solutions

$$\tilde{C}_{2m+1,2n+1}^{AdS} = e^{-i((m+n+\frac{3}{2})t+(n-m)\phi)} \tilde{R} |2m+1\rangle \langle 2n+1| R^{-1}. \quad (3.17)$$

The physical content of the matter fields C, \tilde{C} is contained in the component of the unit operator V_0^1 when expanded in the basis (3.2), while the other components in this expansion are auxiliary fields [10]. The operation of extracting the component of V_0^1 is referred to as taking the trace, and we will denote it by tr , though it should not be confused with the trace operation in the Hilbert space of the harmonic oscillator. The equations of motion (2.3) imply that the physical fields

$$\Phi = \text{tr } C, \quad \tilde{\Phi} = \text{tr } \tilde{C} \quad (3.18)$$

satisfy the Klein-Gordon equation with $m^2 = -3/4$ [10]. One finds that the physical wavefunctions of the solutions (3.16, 3.17) are, up to an inconsequential normalization factor, given by

$$\begin{aligned} \tilde{\Phi}_{2m,2n} &= e^{-i((n+\frac{1}{4})x_++(m+1/4)x_-)} \text{tr} \left(e^{-i\rho(L_1+L_{-1})} |2m\rangle \langle 2n| \right) \\ &\sim e^{-i\omega_- t - il\phi} (\cosh \rho)^{-2h_+} (\tanh \rho)^l {}_2F_1 \left(h_+ + \frac{l - \omega_-}{2}, h_+ + \frac{l + \omega_-}{2}, l + 1, \tanh^2 \rho \right) \\ \tilde{\Phi}_{2m+1,2n+1} &= e^{-i((n+\frac{3}{4})x_++(m+3/4)x_-)} \text{tr} \left(e^{-i\rho(L_1+L_{-1})} |2m+1\rangle \langle 2n+1| \right) \\ &\sim e^{-i\omega_+ t - il\phi} (\cosh \rho)^{-2h_+} (\tanh \rho)^l {}_2F_1 \left(h_+ + \frac{l - \omega_+}{2}, h_+ + \frac{l + \omega_+}{2}, l + 1, \tanh^2 \rho \right) \end{aligned} \quad (3.19)$$

where $h_+ = 3/4$, $h_- = 1/4$, and we have defined

$$l = n - m \quad (3.20)$$

$$\omega_{\pm} = l + 2h_{\pm} + 2m. \quad (3.21)$$

The first identity in the above formulas follows using the cyclicity of tr , while in second identity we evaluated the required traces using (A.11). Comparing to [27], we see that $\Phi_{2m,2n}$ and $\Phi_{2m+1,2n+1}$ are precisely the positive frequency modes of the Klein-Gordon field in the ‘alternate’ and ‘standard’ quantizations respectively, with the correct frequency spectrum to guarantee regularity in global AdS_3 . The solutions

$$\tilde{\Phi}_{0,0} = e^{-it/2}(\cosh \rho)^{-1/2}, \quad \tilde{\Phi}_{1,1} = e^{-3it/2}(\cosh \rho)^{-3/2} \quad (3.22)$$

represent the lowest weight vectors in the decomposition (3.14), and the action of $hs[1/2]$ is realized on the wavefunctions in terms of differential operators. For example, one can show that the generators of the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ subgroup act by Lie derivatives [18]. One can perform a similar analysis for the mode solutions of the field C leading the negative frequency solutions of the Klein-Gordon field in global AdS_3 . In the standard approach to AdS/CFT, one imposes boundary conditions which select only one of the representations in (3.14).

It would be interesting to extend this analysis to the theories based on the higher spin algebra $hs[\lambda]$ which have a realization in terms of deformed oscillators [28]. When λ equals N , a natural number greater than one, the generators V_m^s with $s > N$ form an ideal which can be quotiented out, after which the higher spin algebra becomes $sl(N, \mathbb{R})$ and the associative algebra becomes the algebra of $N \times N$ matrices. In this case the single particle spectrum was analyzed in [18].

4 Matter coupled to the higher spin square

Now we turn to the example of interest, where the higher spin Lie algebra \mathfrak{h} is the higher spin square hss and \mathfrak{a} is the underlying associative algebra \mathfrak{a}_{hss} , both of which we will presently describe following [6]. The algebra \mathfrak{a}_{hss} has an oscillator realization in terms of the chiral modes⁵ of the $\mathcal{N} = 4$ sigma model on T^4 . This free superconformal field theory consists of two complex scalars and two complex fermions with NS boundary conditions, leading to chiral oscillator modes⁶ $a_m^\alpha, \bar{a}_m^\alpha, \psi_m^\alpha, \bar{\psi}_m^\alpha$, with $m \in \mathbb{N}_0, r \in \mathbb{N} + \frac{1}{2}, \alpha \in \{1, 2\}$, and

⁵As in [6], we will ignore the zero modes and work in the sector of zero momentum and winding.

⁶Our conventions are related to those of [29] as follows: starting from the oscillators of four real bosons, b_m^i , and four real fermions, σ_m^i , with $i = 1, \dots, 4$, satisfying

$$\left[b_m^i, \left(b_n^j \right)^\dagger \right] = \delta^{ij} \delta_{mn} 1, \quad \left\{ \sigma_r^i, \left(\sigma_s^j \right)^\dagger \right\} = \delta^{ij} \delta_{rs} 1$$

we have defined

$$\begin{aligned} a_m^1 &= \frac{1}{\sqrt{2}}(b_m^1 + ib_m^2), & \bar{a}_m^1 &= \frac{1}{\sqrt{2}}(b_m^1 - ib_m^2), & a_m^2 &= \frac{1}{\sqrt{2}}(b_m^4 + ib_m^3), & \bar{a}_m^2 &= \frac{1}{\sqrt{2}}(b_m^4 - ib_m^3) \\ \psi_r^1 &= \frac{1}{\sqrt{2}}(\sigma_r^1 + i\sigma_r^2), & \bar{\psi}_r^1 &= \frac{1}{\sqrt{2}}(\sigma_r^1 - i\sigma_r^2), & \psi_r^2 &= \frac{1}{\sqrt{2}}(\sigma_r^4 + i\sigma_r^3), & \bar{\psi}_r^2 &= \frac{1}{\sqrt{2}}(\sigma_r^4 - i\sigma_r^3) \end{aligned}$$

their Hermitian conjugates. The canonical (anti-) commutation relations are

$$\left[a_m^\alpha, \left(a_n^\beta \right)^\dagger \right] = \delta^{\alpha\beta} \delta_{mn} 1, \quad \left[\bar{a}_m^\alpha, \left(\bar{a}_n^\beta \right)^\dagger \right] = \delta^{\alpha\beta} \delta_{mn} 1 \quad (4.1)$$

$$\left\{ \psi_r^\alpha, \left(\psi_s^\beta \right)^\dagger \right\} = \delta^{\alpha\beta} \delta_{rs} 1, \quad \left\{ \bar{\psi}_r^\alpha, \left(\bar{\psi}_s^\beta \right)^\dagger \right\} = \delta^{\alpha\beta} \delta_{rs} 1 \quad (4.2)$$

with all other (anti-) commutators vanishing. The algebra \mathfrak{a}_{hss} is the subalgebra of the oscillator algebra consisting of operators which annihilate both the in- and out Fock vacuum. A basis for \mathfrak{a}_{hss} is formed by the normal-ordered monomials in the oscillators which contain at least one creation and one annihilation operator. The algebra hss is the Lie algebra obtained from \mathfrak{a} by defining the Lie bracket to be the commutator in \mathfrak{a} .

The algebra hss contains a subalgebra $su(1,1|2)$, which arises as the vacuum-preserving subalgebra of the (small) $\mathcal{N} = 4$ superconformal algebra which governs the T^4 theory [29]. The $su(1,1|2)$ algebra consists of $sl(2)$ generators $L_0, L_{\pm 1}$, $su(2)$ R-symmetry generators J_0^i and superconformal generators $G_{\pm \frac{1}{2}}^\alpha, \bar{G}_{\pm \frac{1}{2}}^\alpha$. These are realized in terms of quadratic monomials in the oscillator modes. The expressions for the conformal weight L_0 and R-charge generator J_0^3 are:

$$\begin{aligned} L_0 &= \sum_{n=1}^{\infty} \left[n \left((a_n^\alpha)^\dagger a_n^\alpha + (\bar{a}_n^\alpha)^\dagger \bar{a}_n^\alpha \right) + \left(n - \frac{1}{2} \right) \left(\left(\psi_{n-\frac{1}{2}}^\alpha \right)^\dagger \psi_{n-\frac{1}{2}}^\alpha + \left(\bar{\psi}_{n-\frac{1}{2}}^\alpha \right)^\dagger \bar{\psi}_{n-\frac{1}{2}}^\alpha \right) \right] \\ J_0^3 &= -\frac{1}{2} \sum_{n=1}^{\infty} \left(\left(\psi_{n-\frac{1}{2}}^\alpha \right)^\dagger \psi_{n-\frac{1}{2}}^\alpha - \left(\bar{\psi}_{n-\frac{1}{2}}^\alpha \right)^\dagger \bar{\psi}_{n-\frac{1}{2}}^\alpha \right), \end{aligned} \quad (4.3)$$

where a sum over the index $\alpha = 1, 2$ is implied. Expressions for the other generators can be found in [29].

As in the previous example we would like to work out the decomposition (2.12) of \mathfrak{a}_{hss} into irreducible $HSS \times \widehat{HSS}$ representations, where the two copies act from the right and left respectively. The basis of \mathfrak{a}_{hss} consisting of normal-ordered monomials is not adapted to such a decomposition, since for example L_0 and J_0^3 don't act diagonally either from the left or the right. Once again, a suitable basis for this purpose is the Fock basis, where we expand general operators in basis elements of the form

$$|F_1\rangle \langle F_2|, \quad (4.4)$$

where $|F_1\rangle$ and $|F_2\rangle$ are elements of the Fock space \mathcal{F} built from acting with the creation modes in (4.2) on the vacuum.

The transformation between the normal ordered monomial and Fock bases can be worked out explicitly. For notational simplicity, let us illustrate this for a single bosonic oscillator mode a and for a fermionic mode ψ . For a bosonic oscillator, the relation between normal-ordered and Fock-ordered monomials is summarized by (see Appendix A for more details):

$$\begin{aligned} [(a^\dagger)^m a^n]_F &= : (a^\dagger)^m a^n e^{-a^\dagger a} : \\ (a^\dagger)^m a^n &= [(a^\dagger)^m a^n e^{a^\dagger a}]_F. \end{aligned} \quad (4.5)$$

For a fermionic oscillator, similar formulas hold:

$$\begin{aligned} [(\psi^\dagger)^m \psi^n]_F &= : (\psi^\dagger)^m \psi^n e^{-\psi^\dagger \psi} : \\ (\psi^\dagger)^m \psi^n &= [(\psi^\dagger)^m \psi^n e^{\psi^\dagger \psi}]_F, \end{aligned} \quad (4.6)$$

where most of the terms in the expansion of the right hand side actually vanish due to $(\psi^\dagger)^2 = \psi^2 = 0$. Returning to the full algebra \mathfrak{a}_{hss} , the monomial and Fock operator bases can be worked out in principle by repeated application of (4.5, 4.6). Note that in the Fock basis, both L_0 and J_0^3 act diagonally both from the left and right.

We have not yet imposed, on our Fock operator basis, the restriction that that elements of \mathfrak{a}_{hss} should annihilate the in- and out- Fock vacua. Decomposing the Fock space \mathcal{F} as

$$\mathcal{F} = \mathbb{C} |0\rangle \oplus \mathcal{F}', \quad (4.7)$$

where \mathcal{F}' is the direct sum of the one-, two-, and more-particle Hilbert spaces, a basis for \mathfrak{a}_{hss} is formed by elements of the form

$$|F'_1\rangle \langle F'_2| \quad \text{with } |F'_1\rangle, |F'_2\rangle \in \mathcal{F}'. \quad (4.8)$$

It follows that, as a vector space, \mathfrak{a}_{hss} is simply

$$\mathfrak{a}_{hss} = \mathcal{F}' \otimes \mathcal{F}'^* \quad (4.9)$$

Viewed as representations of hss , the $\mathbb{C} |0\rangle$ term in the decomposition (4.7) corresponds to the the trivial representation, while the second term \mathcal{F}' carries a representation which turns out to be the minimal representation V_{min} discussed in [4],[6]. Indeed, from the above considerations we easily compute the character

$$\begin{aligned} \chi_{\mathcal{F}'} &\equiv \text{tr}_{\mathcal{F}'} q^{L_0} y^{2J_0^3} \\ &= \text{tr}_{\mathcal{F}} q^{L_0} y^{2J_0^3} - 1 \\ &= Z_{T^4} - 1 \end{aligned} \quad (4.10)$$

where Z_{T^4} is the chiral partition function (1.7) of the T^4 SCFT. This agrees with the character of the minimal representation (1.6) of hss [4],[6].

Combining (4.9) with the observation that $\mathcal{F}' = V_{min}$ we arrive at the following simple decomposition (2.12) in the case of the higher spin square:

$$\mathfrak{a}_{hss} = (V_{min}, V_{min}^*). \quad (4.11)$$

The positive frequency solutions which furnish this representation are, from (2.10),

$$\tilde{C}_{F'_1, F'_2} = e^{-i(h_{F'_1} x_- + h_{F'_2} x_+)} \tilde{R} |F'_1\rangle \langle F'_2| R^{-1} \quad (4.12)$$

where h_F denotes the L_0 -eigenvalue of $|F\rangle$. From (2.17) we read off that the partition function of the single particle states of the matter field is

$$Z_{hss}^{1-part}(q, \bar{q}, y, \bar{y}) = \text{tr}_{\mathfrak{a}_{hss}} q^{L_0} \bar{q}^{\bar{L}_0} y^{2J_0^3} \bar{y}^{2\bar{J}_0^3} = |\chi_{min}(q, y)|^2. \quad (4.13)$$

which reproduces (1.5). For completeness, let us also indicate how the multi-particle spectrum, keeping track of statistics, leads to the full CFT result (1.4). We define degeneracies $c(h, \tilde{h}, l, \tilde{l})$ from writing the single-particle partition function as

$$Z_{hss}^{1-part} = \sum_{h, \tilde{h}, l, \tilde{l}} c(h, \tilde{h}, l, \tilde{l}) q^h \bar{q}^{\tilde{h}} y^l \bar{y}^{\tilde{l}}. \quad (4.14)$$

We further note that $c(h, \tilde{h}, l, \tilde{l})$ counts bosonic states when $l + \tilde{l}$ is even and fermions when $l + \tilde{l}$ is odd. Therefore the multiparticle partition function can be written as

$$Z_{hss}^{matter} = \prod_{h, \tilde{h}} \frac{\prod_{l, \tilde{l}; l+\tilde{l} \text{ odd}} (1 + q^h \bar{q}^{\tilde{h}} y^l \bar{y}^{\tilde{l}})^{c(h, \tilde{h}, l, \tilde{l})}}{\prod_{l, \tilde{l}; l+\tilde{l} \text{ even}} (1 - q^h \bar{q}^{\tilde{h}} y^l \bar{y}^{\tilde{l}})^{c(h, \tilde{h}, l, \tilde{l})}}. \quad (4.15)$$

After some algebra this can be rewritten in the form (1.4).

We note that, unlike in the $hs[1/2]$ case (3.14), the decomposition (4.11) contains only one term, so that unlike in that example we don't have a choice of several boundary conditions which are compatible with the hss symmetry. This seems similar to what happens in other examples (see e.g. [30]) where only one of the two possible boundary conditions on a scalar is compatible with supersymmetry.

5 Generalization and outlook

In this note we have confirmed Gaberdiel and Gopakumar's conjecture that the full contribution (1.4) to the untwisted sector CFT partition function arises from a single matter field in the bulk, which is furthermore coupled to the hss gauge fields in a straightforward generalization of Vasiliev's linearized theory [10]. It would be interesting to understand better how this spectrum decomposes in terms of supermultiplets of the AdS supergroup $SU(1, 1|2) \times SU(1, 1|2)$ (see [31]) contained in $HSS \times \widetilde{HSS}$. It would also be of interest to understand how, after imposing appropriate boundary conditions on the gauge fields as in [13], [14], the spectrum of boundary excitations reproduces the extended vacuum character (1.3).

The symmetric orbifold CFT also contains twisted sectors, whose decomposition into $HSS \times \widetilde{HSS}$ representations is not fully understood at present. We note that any representation (R, \tilde{R}) entering in such a decomposition can be described in the bulk by fields C and \tilde{C} taking values in $R \otimes \tilde{R}$ and $\tilde{R} \otimes R$ respectively, with linearized equations

$$\begin{aligned} dC - A_R C + C \tilde{A}_{\tilde{R}} &= 0 \\ d\tilde{C} - \tilde{A}_{\tilde{R}} \tilde{C} + \tilde{C} A_R &= 0 \end{aligned} \quad (5.1)$$

where A_R and $\tilde{A}_{\tilde{R}}$ are the gauge fields in the representations R, \tilde{R} respectively.

These considerations show how the full spectrum of the symmetric orbifold can in principle be reproduced from linearized matter fields in the bulk. The main open question is of course if and how these fields can be incorporated into a consistent interacting theory which furthermore reproduces the correlation functions of the symmetric orbifold. The hope is that the hss symmetry will be sufficiently restrictive that to determine the interactions essentially uniquely, as is the case for the Vasiliev theories.

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A Some ordering identities

In this section we will work out some relations between different ordering prescriptions for operators constructed out of a single harmonic oscillator needed in the main text. In particular, we need to relate the Fock basis to the basis of Weyl-ordered monomials in section 3 and to the basis of normal ordered monomials in section 4. It's useful to define a 'Fock ordering' as the operation which turns monomials in the variables a, a^\dagger into operators by letting the creation and annihilation operators act on $|0\rangle \langle 0|$ from the left and the right respectively, e.g.

$$[(a^\dagger)^m a^n]_F = (a^\dagger)^m |0\rangle \langle 0| a^n. \quad (\text{A.1})$$

Any operator can be expanded in the bases of Fock-, normal- or Weyl-ordered monomials as

$$\mathcal{O} = [f_{\mathcal{O}}^F(a, a^\dagger)]_F =: f_{\mathcal{O}}^N(a, a^\dagger) := [f_{\mathcal{O}}^W(a, a^\dagger)]_W \quad (\text{A.2})$$

where $f_{\mathcal{O}}^F, f_{\mathcal{O}}^N, f_{\mathcal{O}}^W$ are the Fock-, normal- and Weyl-ordered symbols respectively. These are given by

$$f_{\mathcal{O}}^F(z, \bar{z}) = \sum_{m,n \in \mathbb{N}} \frac{\langle m | \mathcal{O} | n \rangle}{\sqrt{m!n!}} \bar{z}^m z^n \quad (\text{A.3})$$

$$f_{\mathcal{O}}^N(z, \bar{z}) = \langle z | \mathcal{O} | z \rangle \quad (\text{A.4})$$

$$f_{\mathcal{O}}^W(z, \bar{z}) = \frac{2e^{2|z|^2}}{\pi} \int d^2w \langle -w | \mathcal{O} | w \rangle e^{-2(w\bar{z} - \bar{w}z)} \quad (\text{A.5})$$

where we have made use of the coherent states

$$|z\rangle = e^{-\frac{1}{2}z\bar{z}} e^{za^\dagger} |0\rangle. \quad (\text{A.6})$$

For a derivation of last two expressions (A.4, A.5) see e.g. [32].

From (A.3, A.4) we find the following simple relation between the normal- and Fock-ordered symbols:

$$\begin{aligned} f_{\mathcal{O}}^F(z, \bar{z}) &= f_{\mathcal{O}}^N(z, \bar{z}) e^{|z|^2} \\ f_{\mathcal{O}}^N(z, \bar{z}) &= f_{\mathcal{O}}^F(z, \bar{z}) e^{-|z|^2}. \end{aligned} \quad (\text{A.7})$$

Applied to monomial basis elements these lead to the expressions (4.5).

Next we wish to derive a relation between Weyl-ordered and Fock-ordered symbols. We start from the expressions relating Weyl- and normal-ordered symbols⁷ (see (5.15) and

⁷These formal expressions are to be treated with caution as convergence issues may arise, see [34] for more details.

(5.30) in [32])

$$\begin{aligned} f_{\mathcal{O}}^N(z, \bar{z}) &= \frac{2}{\pi} \int d^w f_{\mathcal{O}}^W(w, \bar{w}) e^{-2|w-z|^2} \\ f_{\mathcal{O}}^W(z, \bar{z}) &= e^{-\frac{1}{2}\partial_z \partial_{\bar{z}}} f_{\mathcal{O}}^N(z, \bar{z}) \end{aligned} \quad (\text{A.8})$$

Combining these with (A.7) we obtain a relation between Weyl-ordered and Fock-ordered symbols:

$$\begin{aligned} f_{\mathcal{O}}^F(z, \bar{z}) &= \frac{2e^{|z|^2}}{\pi} \int d^2 w f_{\mathcal{O}}^W(w, \bar{w}) e^{-2|w-z|^2} \\ f_{\mathcal{O}}^W(z, \bar{z}) &= e^{-\frac{1}{2}\partial_z \partial_{\bar{z}}} \left(f_{\mathcal{O}}^F(z, \bar{z}) e^{-|z|^2} \right). \end{aligned} \quad (\text{A.9})$$

Application of the first equation leads, after some algebra, to the expansion of Weyl-ordered basis elements in terms of Fock-ordered ones (3.5). For the inverse relation, instead of using the second equation it is more convenient to apply (A.5), leading to (3.6).

From (A.5) we also derive an expression [33] for the trace operation, defined as extracting the coefficient of the unit operator in an expansion in Weyl-ordered monomials,

$$\text{tr } \mathcal{O} = \frac{2}{\pi} \int d^2 z \langle -z | \mathcal{O} | z \rangle. \quad (\text{A.10})$$

From this we compute the following traces needed in section 3:

$$\begin{aligned} \text{tr} \left(e^{-i\rho(L_1+L_{-1})} |m\rangle \langle n| \right) &= \\ \mathcal{N}_{m,n} (\cosh \rho)^{-\frac{3}{2}} (\tanh \rho)^{\frac{n-m}{2}} {}_2F_1 \left(\frac{1-m}{2}, \frac{n}{2} + 1, \frac{n-m}{2} + 1, \tanh^2 \rho \right) \end{aligned} \quad (\text{A.11})$$

where $\mathcal{N}_{m,n}$ is a constant which is irrelevant for our purposes. To derive this result we made use of the BCH-type rearrangement formula

$$e^{i\mu(L_1+L_{-1})} = e^{i \tanh \mu L_{-1}} e^{i \cosh \mu \sinh \mu L_1} (\cosh \mu)^{-2L_0}. \quad (\text{A.12})$$

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